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Asymptotic analysis of crack interaction with free boundary

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Abstract

This paper employs the beam and dipole asymptotic techniques for modelling interaction of a crack with parallel free boundaries. Two configurations are considered: (1) a crack in a half-plane and (2) a crack in the centre of an infinite strip. Both, the stress intensity factors and the areas of the crack opening are calculated.

For the crack situated close to the boundary, the part of the material between the crack and the boundary is represented by a beam (plate in plane-strain). This allows calculating the area of the crack opening. The stress intensity factors are calculated by matching the beam approximation with Zlatin and Khrapkov's solution (Zlatin and Khrapkov, 1986) for a semi-infinite crack parallel to the boundary of a half-plane or with Entov and Salganik's solution (Entov and Salganik, 1965) for a central semi-infinite crack in a strip. It has been shown that this asymptotic method allows obtaining two leading terms for the SIFs and the crack opening area.

When the distance between the crack and the free surface is large, the problem is treated in the far field approximation. This, dipole asymptotic method allows finding the leading asymptotic terms responsible for the crack–boundary interaction.

For intermediate distances between the crack and the boundary, simple interpolating formulas are derived. Particular examples of cracks loaded by pair of concentrated forces and for uniform loading are considered. The obtained results are compared with available numerical solutions. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Beam approximation; Dipole asymptotics; Stress intensity factors; Area of crack opening

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1. Introduction

Crack growth in the presence of a parallel rectilinear boundary is an important situation for various applications. Thus, the interaction between the crack and the boundary underlies some mechanisms of skin rock burst (e.g., Fairhurst and Cook, 1966; Dyskin and Germanovich, 1993), bore-hole breakouts (Ewy and Cook, 1990; Germanovich et al., 1994), and rock thermal spallation under the elevated temperatures (e.g., Rauenzahn and Tester, 1989; Germanovich and Goncharov, 1997). This type of interaction can also occur in some types of fracture toughness measurements (Obreimoff, 1930; Lawn, 1993), volcanic eruptions (Germanovich and Lowell, 1995) and brittle cutting of materials and rocks (Ingraffea and Heuze, 1980; Cherepanov et al., 1987; Dyskin et al., 1994a).

At present, the full solution for such problems can be obtained only numerically. This can be done via such direct descritezation techniques as finite and boundary element methods (e.g., Lardner et al., 1990) or by solving numerically a system of singular integral equations (e.g., Erdogan et al., 1973). Another approach could be in using a solution for a dislocation in half-space (e.g., Weertman, 1996) as the Green function. However, if the crack is situated too close to the boundary, numerical solutions can lose the accuracy requiring significant adjustments of the employed algorithm (e.g., Germanovich and Grekov, 1998). On the other hand, integrating the Green function, in this particular case, represents a difficult numerical task by itself because it requires a non-obvious regularization of singular integrals in infinite limits prior to their computing (e.g., see example provided by Weertman, 1996). Murakami (1987) presents two numerical solutions for the case of uniformly loaded crack in a half-plane parallel to the free boundary. For long cracks, these solutions differ by more than 40%. Therefore, there is a need for asymptotic solutions that are free of such shortcomings and can serve as reference points for verifying one or another numerical scheme. In addition, the asymptotic solutions usually provide relatively simple analytical expressions, which are very useful when practical multi-parameter problems are studied. Furthermore, it is much easier to analyze generic properties of multi-parameter crack systems in the vicinities of singular points (which are normally of the main interest) in the space of controlling parameters if analytical expressions describing the crack behavior are available (Germanovich and Cherepanov, 1995).

The problem for interaction between a crack and a free boundary has two obvious extremes.

1. If the crack is situated far from the boundary, i.e., $l/h \ll 1$, where l is the crack half-length and h is the distance between the crack and free boundary, the problem can be solved in power series with respect to this ratio (e.g., Savruk, 1981; Chudnovsky and Kachanov, 1983). In this case, the main term is provided by the solution for the crack in an unbounded body. The next term is the first one accounting for the influence of the boundary. This term can be found by the method of dipole asymptotics (e.g., Gol'dstein and Kaptsov, 1982; Kachanov, 1987; Dyskin et al., 1992; Dyskin and Mühlhaus, 1995). The method is based on considering the stress disturbance produced by the crack in an unbounded body and keeping only the first terms of the order of l^2/h^2 (in the 2-D case) which constitute the far-field approximation.
2. If the crack is situated close to the boundary, i.e., $l/h \gg 1$, the material between the crack and the boundary can be considered as a beam under the given load (e.g., Rice, 1968a, 1968b; Slepian, 1981; Williams, 1988; Bolotin, 1996). This representation makes it possible to calculate the change, G, of the elastic energy due to an infinitesimal step of the crack propagation. Since $G \propto K_I^2 + K_{II}^2$, where K_I and K_{II} are the stress intensity factors (SIFs) of modes I (crack under pure tension) and II (crack under transverse shear), respectively, this energy approach, in general, does not allow separating the SIFs. In particular, Williams (1988) attempted to partition the modes by considering a strip with a parallel crack and assuming that loading along the crack does not produce mode I stress concentration and tearing does not produce mode II stress concentration. This is correct for a

symmetrically loaded central crack but not in general case (e.g., see the numerical solution by Grekov et al., 1992). Therefore, Williams (1988) assumption, in fact, raises an important and general issue of determining for what loading a separation of fracture modes is possible. This motivated Zlatin and Khrapkov (1986, 1990) to solve the problem for a half-infinite crack parallel to the boundary of a half-plane expressing the SIFs through the moment and total force of the applied load. Nazarov and Polakova (1990), addressed a similar question for parallel cracks lying close together in a plane region. Their results confirm that normal and shear tractions applied separately on crack sides generally produces both mode I and mode II SIFs.

In this paper, the results of our preliminary conference publications (Dyskin and Germanovich, 1993; Dyskin et al., 1993; 1994b; Ustinov et al., 1994; Germanovich, 1997) are further developed and both asymptotic cases ($l/h \gg 1$ and $l/h \ll 1$) are studied in conjunction with the problems for a crack parallel to a half-plane boundary and a central crack in a strip. Simple approximate formulae for SIFs and the areas of crack openings are obtained for all considered cases by interpolation between two asymptotic solutions: for cracks close to and far from the boundary. The interpolation formulae are verified against available numerical solutions.

2. Cracks close to a free boundary. Beam asymptotic approximation

Consider a 2-D elastic problem for a *finite* crack parallel to the boundary of a semi-infinite isotropic plane [Fig. 1(a)]. Let the distance, h, to the boundary be much smaller than the crack length, $2l$. The goal is to calculate SIFs, energy release rate and crack opening. The method proposed in this section is based on matching inner and outer asymptotics and allows for calculating two leading asymptotic terms. Since $l \gg h$, the solution for the *inner* region surrounding the crack tip can be obtained by considering a semi-infinite crack [Fig. 1(c)]. This representation is valid for all the points situated far enough from the other tip of the crack or, in other words, for the crack points, x, such that $0 \leq x \ll l$ where x is the coordinate of a crack point and coordinate set, Oxy, has its origin, O, at the left tip of the crack [Fig. 1(c)]. The solution for the *outer* region can be found by employing the beam approximation [Fig. 1(b)].

Therefore, the first step could be to directly use Zlatin and Khrapkov's solution for a semi-infinite

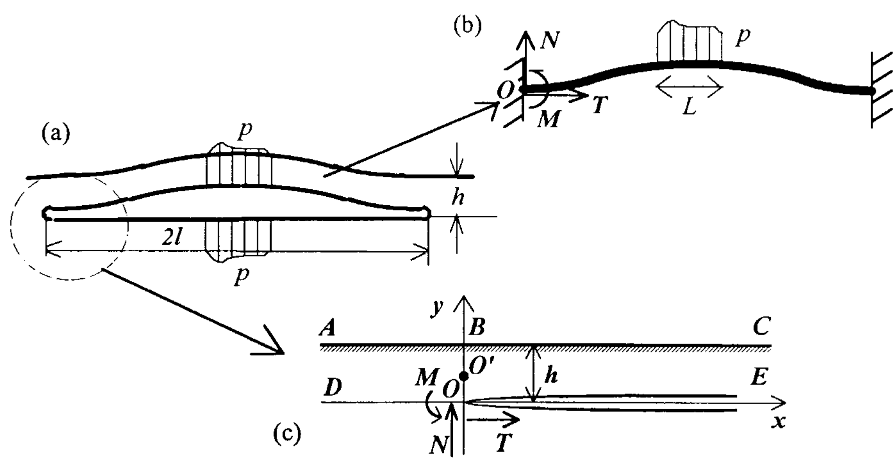

Fig. 1. The crack close to boundary: (a) the original problem (real crack); (b) the outer problem (beam asymptotic approximation); and (c) the inner problem (semi-infinite crack).

crack shown in Fig. 1(c) (inner region). However, this solution addresses only the case when neither the crack nor the surface of the half-space are loaded. In other words, Zlatin and Khrapkov found a non-trivial solution of a homogeneous problem and expressed the SIFs through the total moment and force of the stresses acting on the line of the crack continuation. Fortunately, as shown in Appendix A, in the case when the faces of the semi-infinite crack are loaded by arbitrary load distributions bounded near the crack tip and with finite total force and moment (Fig. 1), this solution permits finding two successive asymptotic terms with respect to the small parameter, h/L, where L is the typical scale of the variation of the surface traction [Fig. 1(b)]. Furthermore, for a wide range of loads, their moment about the crack tip determines the first, leading term of the asymptotics while the total force contributes only to the second-order term.

The second step is in considering the thin layer of material between the finite crack and the boundary [Fig. 1(a)] as a beam [Fig. 1(b)]. This allows obtaining the total force and moment about the beam end (crack tip) while the expression for the displacement (deflection) of the beam gives an asymptotic approximation for displacements of the crack surfaces with respect to the small parameter, h/l. This is valid for all the points, x, of the crack located far from its ends compared to h ($2l \geq x \gg h$, $2l - x \gg h$), which gives a solution for the *outer* region of the considered problem.

The third step is in matching these two asymptotics; the matching region consisting of the points, x, such that $h \ll x \ll l$. Below, the matching of the asymptotics is considered in detail.

In order to use the elementary beam solution for the determination of the total force and moment, the end conditions are necessary. It is shown below that the conventional assumption of ideal clamping of the beam ends (e.g., Slepian, 1981) provides only the first leading term for the SIFs. To correctly obtain the second term, one has to suppose the condition of elastic clamping. The unknown coefficient of elastic clamping will be determined by comparing the energy release rates computed independently, i.e., by using:

1. the elementary beam solution and
2. the SIFs obtained by Zlatin and Khrapkov (1986, 1990).

2.1. Inner problem

Let the semi-infinite crack occupy the part $x > 0$ of the x-axis [Fig. 1(c)]. The problem for such a crack has been solved by Zlatin and Khrapkov (1986, 1990) who supposed that both the crack faces and the half-plane boundary are free of load (homogeneous problem). They have found the eigen solutions for this homogeneous problems satisfying the following conditions for stress distributions on crack continuation, $x < 0$ [in the coordinate system shown in Fig. 1(c)]:

$$M = \int_{-\infty}^{0} x\sigma_{yy}(x,0)\mathrm{d}x,$$

$$N = \int_{-\infty}^{0} \sigma_{yy}(x,0)\mathrm{d}x$$

$$T = \int_{-\infty}^{0} \sigma_{xy}(x,0)\mathrm{d}x, \tag{1}$$

where σ_{xy} and σ_y are the stress components; the convergence of the integrals in Eq. (1) is presumed.

From a physical standpoint, (N, T) and M correspond to the total forces and moment applied at infinity ($x \to +\infty$) to the strip $0 \le y \le h$ of material above the crack. Indeed, they should be balanced by the stresses acting on the line of the crack continuation (since the strip $0 \le y \le h$, $-\infty < x < +\infty$ is in equilibrium) resulting in Eq. (1). Following Zlatin and Khrapkov (1986, 1990), these forces and moments are shown in Fig. 1(c) near the crack tip and are supposed to be known.

According to Zlatin and Khrapkov (1986, 1990), the expressions for the SIFs, K_{I} and K_{II}, and the energy release rate, $G = \mathrm{d}U/\mathrm{d}l$, associated with an infinitesimal step of the crack propagation have the following form:

$$\begin{pmatrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{pmatrix} = K_M M h^{-3/2} + K_N N h^{-1/2} - \left(K_T - \frac{1}{2} K_M \right) T h^{-1/2}, \quad (2)$$

$$\frac{\mathrm{d}U}{\mathrm{d}l} = \frac{1}{Eh^3}\left[6\left(M - \frac{1}{2} Th + \delta \cdot Nh \right)^2 + \frac{h^2}{2}(T + \sqrt{3}N)^2 \right], \quad (3)$$

where factors

$$K_M = \begin{pmatrix} 1.932 \\ 1.506 \end{pmatrix},$$

$$K_N = \begin{pmatrix} 1.951 \\ -0.032 \end{pmatrix},$$

$$K_T = \begin{pmatrix} 0.4346 \\ -0.5578 \end{pmatrix}$$

$$\delta = 0.620. \quad (4)$$

Hereafter, E is Young's modulus and all the formulae are written for the case of plane stress; for the plane strain, E should be replaced, as usual, by $E/(1-\nu^2)$ where ν is Poisson's ratio.

It should be noted that, according to the *exact* solution Eq. (2), the crack is under mixed-mode conditions even if the applied load is pure tensile or shear.

Zlatin and Khrapkov (1986, 1990) assumed the crack faces and half-plane boundary to be free of load. Consider now an arbitrary load distributed over the crack faces [Fig. 1(a)] while the half-plane boundary is still free of tractions (otherwise, the problem could be decomposed into two: loaded half-plane without the crack and a loaded crack in a half-plane with non-loaded boundary). Let (N, T) and M be the total force and moment of this load, respectively, and the characteristic size, L, associated with the loading [Fig. 1(b)] be comparable to the crack dimension, $L \sim l \gg h$ [Fig. 1(c)]. Then $M \sim Nl = Nh\varepsilon^{-1}$, where $\varepsilon = h/l$, and Eq. (2) formally gives two leading consecutive terms, $O(\varepsilon^{-1})$ and $O(1)$, with respect to ε ($\varepsilon \to 0$, h = const). Appendix A shows that Eqs. (2), (3) and (4) can indeed be used for arbitrary loads and, furthermore, the load with zero total force and moment contributes only to terms of higher order. In other words, the difference between two loads having the same total force and moment can and will be ignored in the following consideration.

2.2. Outer problem

In order to relate the total force, (N, T), and the moment, M, to the *external* load acting on the *finite* crack [Fig. 1(a)], the equilibrium of the semi-strip, $ABOD$ ($x < 0$, $0 < y < h$), shown in Fig. 1(c) has to be considered. Since the boundary of the semi-plane is free, the total force, (N, T), and the moment, M, have to be in equilibrium with the total force and the moment of the stresses, acting on line OB. They can be determined by considering the layer between the crack and boundary as a beam (plate) and using the corresponding solutions of the elementary beam theory. This will give two main asymptotic terms for (N, T) and M with respect to $\varepsilon = h/l$ (e.g., Timoshenko and Goodier, 1970).

The outer (beam) approximation is only needed to determine the corresponding total moment, M, and force, N, to be used in Eqs. (2) and (3). Due to the condition of equilibrium, they are 'transmitted' from the outer scale to the inner one, resulting in stress concentration around the crack tip. Therefore, the outer problem is in calculating the deflection of the beam (or plate in the plane strain) with certain end conditions yet to be defined.

It should be noted that in the beam (plate) theory, forces and moments are associated with the neutral line of the beam [the line passing through point O' in Fig. 1(c)] rather than the crack face, OE. While this does not affect the calculations involving the normal component of the load (since it creates the same moment about points O and O'), the replacement of T in the above formulation with the total shear force, T_b, acting along the neutral line will cause an additional moment, $\Delta M \sim T_b h$ about point O. Then this moment should be added to the moment M in (2.2) and (2.3). For a beam with a symmetrical cross section this moment is $\Delta M = T_b h/2$.

Let us first account only for the normal component, $q(x)$, of the load applied to the beam. The differential equation for such a beam has the form (e.g., Landau and Lifshitz, 1959)

$$u'''' = \frac{q(x)}{EI}, \tag{5}$$

where the beam deflection, $u(x)$, is the vertical displacement of the beam and EI is the flexural rigidity. Being related to the unit length in the direction perpendicular to the drawing plane [Fig. 1(b)], the flexural rigidity is

$$EI = \frac{Eh^2}{12}. \tag{6}$$

The general solution of Eq. (5) is

$$u = A + Bx + Cx^2 + Dx^3 + \tilde{u}(x), \tag{7}$$

where A, B, C and D are arbitrary constants and $\tilde{u}(x)$ is a particular solution of Eq. (5), determined by the external load, $q(x)$. We chose $\tilde{u}(x)$ such that its first four derivatives are zero at $x = 0$. Of course, in this case, boundary conditions for another end, $x = 2l$, should not be assigned for $\tilde{u}(x)$.

To determine A, B, C and D, four equations are required and, naturally, one may want to use four boundary conditions for $u(x)$ specified at the beam ends. However, the choice of boundary conditions at the beam ends is not obvious *a priori* because, in reality, the beam under considerations [Fig. 1(b)] represents the layer between the crack and the boundary [Fig. 1(a)].

In principle, the boundary conditions can be written as

$$u|_{x=0} = 0,$$

$$u|_{x=2l} = 0,$$

$$u'|_{x=0} = \kappa EIu''|_{x=0}$$

$$u'|_{x=2l} = \kappa EIu''|_{x=2l}. \tag{8}$$

The first two conditions in Eq. (8) express the displacement continuity at the beam ends. The other two represent the condition of elastic clamping for the left ($x = 0$) and right ($x = 2l$) ends of the beam (κ is the coefficient of elastic clamping). This coefficient will further be determined by matching the outer and inner asymptotics.

The moment and shear force distributions along the beam are given by the well known formulae (e.g., Landau and Lifshitz, 1959)

$$M(x) = EI \cdot u''(x)$$

$$N(x) = -EI \cdot u'''(x). \tag{9}$$

The first condition in Eq. (8) suggests that $A = 0$. Using Eq. (7), the constants C and D can be expressed in terms of the shear force and moment acting at the beam end, $x = 0$:

$$C = \frac{M(0)}{2EI}$$

$$D = -\frac{N(0)}{6EI}. \tag{10}$$

The constant B is the angle of the beam rotation about point O [Fig. 1(b)]. In the general case, the angles of rotation at the beam ends only depends upon the bending moment of the beam at those points, while shear forces does not affect the angles[1]. This dependence is linear because the beam is meant to represent the outer asymptotics for a problem which is linearly elastic. Generally, the coefficient of proportionality is a function of elastic modulus and geometry of the beam cross section. Therefore, the condition of elastic clamping shall be presumed: the angle is proportional to the bending moment with an (unknown) coefficient depending on elastic, E, and geometric, h, parameters.

Since the theory of elasticity is a local theory, in the beam approximation, the coefficient of elastic clamping, κ, should be independent of the beam length, $2l$. The dimension analysis then gives:

$$B = \kappa M(0) = k\frac{1}{h^2}\frac{M(0)}{E} = k\frac{h}{12}u''(0), \tag{11}$$

where k is a dimensionless constant.

Thus, three parameters, $M(0)$, $N(0)$ and k can be expressed through three constants, B, C and D. To determine these constants, Eqs. (7) and (11) should be substituted into Eq. (9), which gives $A = 0$ and

[1] Indeed, due to the linearity of Hook's law, the total force and moment at a point x asymptotically representing stresses in the beam approximation are linear combinations of $u(x)$ and $u'(x)$. Since, at the beam ends, $u(0) = u(2l) = 0$, resolving these linear combinations with respect to angles $u'(0)$ or $u'(2l)$ immediately shows that the angles are proportional to the corresponding moments.

$$\begin{cases} 2Bl + 4Cl^2 + 8Dl^3 + \tilde{u}(2l) = 0 \\ B = \dfrac{k}{6}hC \\ B + 4Cl + 12Dl^2 + \tilde{u}'(2l) = -k\left[\dfrac{1}{6}hC + Dlh + \dfrac{1}{12}\tilde{u}''(2l)h\right] \end{cases} . \tag{12}$$

By introducing a small parameter, $\varepsilon = h/l$, the solution of Eq. (12) can be written in the following form

$$\begin{pmatrix} 2l & 4l & 8l \\ 1 & -\dfrac{k}{6}\varepsilon & 0 \\ 1 & 4+\dfrac{k}{6}\varepsilon & 12+k\varepsilon \end{pmatrix} \times \begin{pmatrix} B \\ Cl \\ Dl^2 \end{pmatrix} = \begin{pmatrix} -\tilde{u}(2l) \\ 0 \\ -\tilde{u}(2l) - \dfrac{k}{12}\tilde{u}''(2l)l\varepsilon \end{pmatrix} \tag{13}$$

$$\begin{pmatrix} B \\ Cl \\ Dl^2 \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 & 0 \\ \dfrac{3}{4l} & -1 & -\dfrac{1}{2} \\ -\dfrac{1}{4l} & \dfrac{1}{4} & \dfrac{1}{4} \end{pmatrix} + k\varepsilon \begin{pmatrix} \dfrac{1}{8l} & -\dfrac{1}{6} & -\dfrac{1}{12} \\ \dfrac{5}{16l} & \dfrac{5}{24} & \dfrac{1}{6} \\ \dfrac{1}{16l} & -\dfrac{1}{16} & -\dfrac{1}{16} \end{pmatrix} + O(\varepsilon^2)\right] \times \begin{pmatrix} -\tilde{u}(2l) \\ 0 \\ -\tilde{u}(2l) - \dfrac{k}{12}\tilde{u}''(2l)l\varepsilon \end{pmatrix}. \tag{14}$$

As follows from Eq. (1), $\tilde{u}(x) = 12E^{-1}h^{-3}f(x,l)$, where the function $f(x,l)$ depends on the applied load, but is independent of h. The dimensional analysis then suggests that $\tilde{u}(x) = 12E^{-1}h^{-3}l^4 f(x/l)$. Therefore, $\tilde{u}'(2l) \propto \tilde{u}(2l)/l$ and $\tilde{u}''(2l) \propto \tilde{u}'(2l)/l$. The coefficients of proportionality in these relationships cannot depend upon the applied load because of the linearity of the elastic beam problems. At the same time, they do not depend upon h. Therefore, all terms containing k are of the order of ε or higher. Substitution of Eq. (14) into Eq. (9) and accounting for Eq. (11) gives

$$M(0) = 2CEI = \frac{Eh^3}{12}\left\{\left[\frac{\tilde{u}'(2l)}{l} - \frac{3}{2}\frac{\tilde{u}(2l)}{l^2}\right] + k\varepsilon\left[\frac{3}{8}\frac{\tilde{u}(2l)}{l^2} - \frac{\tilde{u}'(2l)}{3l} + \frac{1}{12}\tilde{u}''(2l)\right]\right\} + O(\varepsilon^2)$$

$$N(0) = -6DEI = \frac{Eh^3}{12}\left\{\left[\frac{3}{2}\frac{\tilde{u}(2l)}{l^3} - \frac{3}{2}\frac{\tilde{u}'(2l)}{l^2}\right]\right\} + O(\varepsilon^2). \tag{15}$$

Note that $\tilde{u}(x)$, $\tilde{u}'(x)$, $\tilde{u}''(x) \propto h^{-3}$.

It is seen from Eq. (15) that the coefficient of elastic clamping only contributes to the second asymptotic term. Thus, if only the leading terms are sought the coefficient k can be set to zero, which corresponds to the condition of ideal clamping. For the determination of the second term, this coefficient has yet to be found, which will be done in the next section.

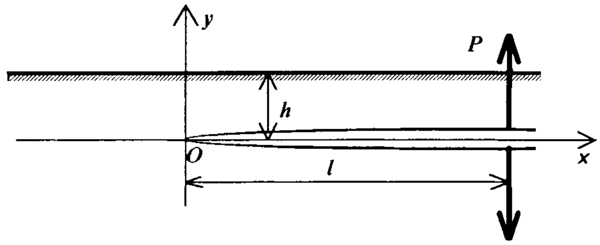

Fig. 2. Simple configuration for determining the elastic clamping constant.

2.3. Matching inner and outer asymptotics

The next step is in matching inner and outer asymptotics. The only free parameter which is left is the constant, k. This constant will be determined by comparing the energy release rates in the outer and inner asymptotics.

Since k characterizes the clamped end of the beam, it does not depend on a particular load distribution and can be determined from any type of loading. Consider the simplest one: the normal force P acting at a distance l from the clamping point O of a semi-infinite beam[2] (Fig. 2), while the rest of the beam is free. This represents a semi-infinite crack loaded by a pair of concentrated forces, P, at a distance l from the tip. Then $M(0)=Pl$, $N(0)=P$, $T=0$.

Considering Eq. (7), the bending energy of such a beam can be written as

$$U=\frac{1}{2}Pu(l)=\frac{P}{2}[Bl+Cl^2+Dl^3]. \tag{16}$$

By inserting Eqs. (10) and (11) into Eq. (16), the energy can be rewritten in the following form:

$$U=\frac{6P^2}{Eh^3}\left[\frac{l^3}{3}+\frac{khl^2}{12}\right]. \tag{17}$$

Then the energy release rate is (see also Rice, 1968a, 1968b):

$$\frac{\mathrm{d}U}{\mathrm{d}l}=\frac{P^2l^2}{Eh^3}\left[6+k\frac{h}{l}\right]. \tag{18}$$

This energy release rate should coincide with the one computed from considering the stress state at the vicinity of the crack tip[3], i.e., with Eq. (3). The comparison of the first two asymptotic terms gives

$$k=12\delta=7.440. \tag{19}$$

Thus, the elastic clamping constant has been determined for the beam representing the layer between the crack and free boundary.

Therefore, in order to calculate the stress intensity factors for a crack parallel to a free rectilinear boundary, it is necessary to consider the corresponding beam under the conditions of elastic clamping, coefficient being $\kappa=k/(Eh^2)=12\delta/(Eh^2)=\delta h/(EI)$, and calculate the bending moment and total force at

[2] In the vicinity of the end where the matching is conducted, the beam can be considered as semi-infifnite.

[3] Strictly speaking, the energy release rates computed from the elementary beam solution and from the SIFs belong to different scales, greater and less than h, respectively. However, as shown in Appendix B, their matching is still possible with the adopted accuracy.

its end. It is seen from Eq. (15) that only the second term in the expansion for M, i.e. the term of the order of ε, depends on k. This means that if only the main asymptotic term of the SIF is sought, coefficient k may be chosen arbitrarily, e.g., set to zero, which is the pure clamping state. Accordingly, the main asymptotic term for the SIF is

$$\begin{pmatrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{pmatrix} = K_M M_0 h^{-3/2}, \tag{20}$$

where M_0 is determined from the problem for a purely clamped beam ($k = 0$).

The area of the crack opening can be calculated by integrating the vertical displacements of the upper and the lower faces of the crack. The contribution of the displacement of the upper face can be calculated by integrating the beam deflection Eq. (7):

$$S = \int_0^l u(x)\mathrm{d}x. \tag{21}$$

Since the expression for beam deflection Eq. (8) should, in general, contain a term x^3, the integral of Eq. (21) would be of the order of l^4. As demonstrated in Appendix C, Eq. (C1), the displacement of the lower face would, at most, be of the order of $l^2 \ln(l/h)$, which gives two orders less contribution into the area than the upper face one. Therefore with the adopted accuracy, only the area produced by the beam deflection has to be taken into account.

Finally, to calculate two asymptotic terms of area of the crack opening, it is necessary to calculate the integral of deflection of the beam with the above-determined condition of elastic clamping. Similarly to the case of SIFs, to calculate the main asymptotic term of the area of the crack opening, it is sufficient to calculate the integral of the beam deflection under the condition of pure clamping.

When the crack faces are subjected to shear loading, it should be noted that it does not contribute to the moment about the crack tip, while the total shear force, T, contributes only to the second asymptotic term. Hence, the shear force acting at the crack ends can be determined without considering beam deflections, e.g., directly from the shear traction distribution.

Note that Eq. (15) is mainly needed for analysis of the orders (with respect to ε) of the terms in Eqs. (4) and (21). In many cases, in order to find the moment and force, it is easier to solve the corresponding beam problem assuming elastic clamping at its ends rather than use Eq. (15). In the case of symmetrically loaded beam though, Eq. (15) allows us to express the terms of both orders (~ 1 and $\sim\varepsilon$) through the moment and force obtained for pure clamping.

2.4. Symmetrically loaded beam

If the applied load is symmetrical over the mid-point of the crack, then instead of the last equation in Eq. (12), the following condition can be used

$$u'(l) = 0. \tag{22}$$

Similarly to Eq. (15), the solution of the corresponding system in this case is

$$M(0) = 2CEI = EI\left[\frac{\tilde{u}'(2l)}{l} - \frac{3}{2}\frac{\tilde{u}(2l)}{l^2}\right](1 - \delta\varepsilon) + O(\varepsilon^2) = M_0(1 - \delta\varepsilon) + O(\varepsilon^2)$$

$$N(0) = -6DEI = EI\left[\frac{3}{2}\frac{\tilde{u}(2l)}{l^3} - \frac{3}{2}\frac{\tilde{u}'(2l)}{l^2}\right] + O(\varepsilon^2) = N_0 + O(\varepsilon^2). \tag{23}$$

Here M_0 and N_0 are the bending moment and shear force for the corresponding purely clamped beam ($k = 0$). Also, the expression for D not containing k coincides with the one for the purely clamped beam. The expression for $M(0)$ in Eq. (23) differs from the corresponding expression for the purely clamped beam by a factor of $(1-\varepsilon\delta)$. Therefore, in order to calculate two main asymptotic terms of the SIF for a symmetrically loaded beam the following formula can be used:

$$\begin{pmatrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{pmatrix} = K_M M_0 (1-\delta\varepsilon) h^{-3/2} + K_N N_0 h^{-1/2} - \left(K_T - \frac{1}{2} K_M \right) T h^{-1/2}. \tag{24}$$

By substituting Eq. (14) into Eq. (8) with the aid of Eq. (10), the expression for the area of the crack opening can be obtained:

$$S = 2\int_0^l (Bx + Cx^2 + Dx^3 + \tilde{u}(x))\mathrm{d}x = S_0 + \frac{8\delta M_0 l^2}{Eh^2}. \tag{25}$$

Here, S_0 is the area calculated using the pure clamping condition.

2.5. Examples of symmetrical beams. Pair of concentrated forces and uniform load

For a crack of length $2l$ parallel to a free rectilinear boundary situated at a distance $h \ll l$ apart from it, two types of loading will be considered: a pair of concentrated forces, P [Fig. 3(a)] and a uniform stress distribution, p [Fig. 3(b)].

According to the above method, the layer between the crack and the surface has to be treated as a clamped beam, and the bending moment and deflection have to be calculated. For the problem from Fig. 3(a), the deflection of the part $(0, l)$ of the beam and the moment M_0 are (e.g. Landau and Lifshitz, 1959):

$$u = \frac{P}{2Eh^3} x^2 (3l - 2x)$$

$$M_0 = \frac{Pl}{4}. \tag{26}$$

As before, only plane stress is considered; the plane-strain expressions can be obtained by replacing Young's modulus, E, with $E(1-\nu^2)$, where ν is Poisson's ratio.

By substituting Eq. (26) into Eqs. (2) and (4), the asymptotics for both modes of the SIF can be found:

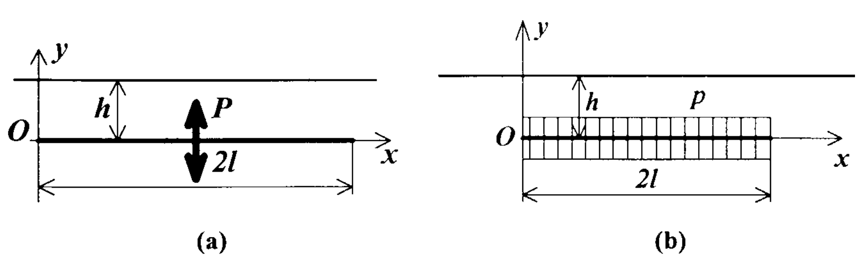

Fig. 3. Beam under a pair of concentrated forces at the centre (a); and under uniform loading (b).

$$K_{\rm I} = P\frac{l}{h^{3/2}}(0.483 + 0.67\varepsilon + O(\varepsilon^2))$$

$$K_{\rm II} = 0.3765P\frac{l}{h^{3/2}}(1 + 0.249\varepsilon + o(\varepsilon)). \tag{27}$$

The leading asymptotic terms in Eq. (27) can be reduced to the ones obtained by Nazarov and Polakova (1990) after correcting a misprint in that paper.

The integration of the beam deflection Eq. (26) and substitution of the result, S_0, into Eq. (25) gives

$$S = Pl\left[\frac{1}{2E}\varepsilon^{-3} + \frac{1.240}{E}\varepsilon^{-2}\right]. \tag{28}$$

For the case of uniform loading, the deflection of the part (0, l) of the beam and the bending moment are

$$u = \frac{P}{2Eh^3}x^2(2l - x)^2$$

$$M_0 = \frac{pl^2}{3}. \tag{29}$$

Accordingly, the SIFs and the area of the crack opening are:

$$K_{\rm I} = 0.644\, p\frac{l^2}{h^{3/2}}(1 + 1.551\varepsilon); \quad K_{\rm II} = 0.502p\frac{l^2}{h^{3/2}}(1 + 1.534\varepsilon)$$

$$S = pl^2\left[\frac{8}{15E}\varepsilon^{-3} + \frac{1.653}{E}\varepsilon^{-2}\right]. \tag{30}$$

Comparison with the numerical results will be described in Section 4.

3. Dipole asymptotics

This method allows calculation of the interaction between the crack and the free surface when the distance from the surface is much greater than the crack size. In general, the interaction can be calculated by introducing an additional stress (not necessarily uniform) distributed over the crack length. This additional stress is the stress produced by the crack and 'reflected' from the boundary (the exact sense of the reflection will be specified later). In the 2-D case, the crack-produced stress disturbance vanishes at infinity as $(l/x)^2$, where $2l$ is the crack length; the reflected field has the same order. If the crack is situated at a distance $h \gg l$ from the boundary, the main term of this additional field on the crack line has the order $(l/h)^2$, while its variation within the crack length is of the next order, $(l/h)^3$. Therefore, to obtain the first asymptotic term, it is sufficient to assume this additional stress to be uniform and equal to the stress that would be produced in the original material without the crack at the location of its centre. Moreover, it is sufficient to calculate this additional stress with the same accuracy, $(l/h)^2$. Then this field formally coincides with the stress field produced by a proper combination of force dipoles situated in the solid material at the place of the crack center (e.g., see details and the history of the method in the papers by Dyskin et al., 1992 or Dyskin and Mühlhaus, 1995). It is also analogous to the field generated by a pair of opposite sign dislocations separated by some distance and often referred to as dislocation dipoles (e.g., Weertman, 1996). That is why this method is called the dipole asymptotic method.

Let us consider a half-plane with a crack of length $2l$ situated at a distance $h \gg l$ from the boundary (Fig. 4) and introduce the complex variable $z = x + \mathrm{i}y$. Let $z_0 = -\mathrm{i}h$ be the coordinate of the crack center. It is convenient to employ Muskhelishvili's complex potentials, $\Phi(z)$ and $\Psi(z)$ (Muskhelishvili, 1963). Then the corresponding stresses can be expressed as

$$\sigma_{xx} + \sigma_{yy} = 2[\Phi(z) + \overline{\Phi(z)}]$$

$$\sigma_{yy} - \sigma_{xx} + 2\mathrm{i}\tau_{xy} = 2[\bar{z}\Phi'(z) + \Psi(z)]. \tag{31}$$

According to the outlined method, the first step in obtaining the dipole asymptotics is the calculation of the remote supplementary stress field generated by the crack in an infinite plane. The complex potentials of this field have the form

$$\Delta\Phi(z) = \frac{1}{2\pi\mathrm{i}\sqrt{z^2 - l^2}} \int_{-l}^{l} \frac{\sqrt{t^2 - l^2}[\sigma_0(t) + \mathrm{i}\tau_0(t)]}{t - z}\,\mathrm{d}t$$

$$\Delta\Psi(z) = \Delta\overline{\Phi(\bar{z})} - \Delta\Phi(z) - z\Delta\Phi'(z), \tag{32}$$

where $\sigma_0(t)$ and $\tau_0(t)$ are the external tractions applied to the crack faces. The asymptotic form of this field in the case when the crack is situated at point $z_0 = -\mathrm{i}h$ is

$$\Delta\Phi(z) = \frac{D_1}{(z + \mathrm{i}h)^2} + O\left(\frac{l^3}{h^3}\right)$$

$$\Delta\Psi(z) = \frac{D_2}{(z + \mathrm{i}h)^2} + \frac{2\mathrm{i}hD_1}{(z + \mathrm{i}h)^3} + O\left(\frac{l^3}{h^3}\right), \tag{33}$$

where the complex constants D_1 and D_2 have the units of moment (per unit length in the direction normal to the plane of drawing) and characterise the dipole moments of the force dipoles representing the crack. They can be expressed as follows

$$D_1 = \frac{\mathrm{i}}{2\pi}\int_{-l}^{l} \sqrt{t^2 - l^2}[\sigma_0(t) + \mathrm{i}\tau_0(t)]\mathrm{d}t = \frac{\mathrm{i}S}{8\pi}$$

$$D_2 = D_1 + \bar{D}_1. \tag{34}$$

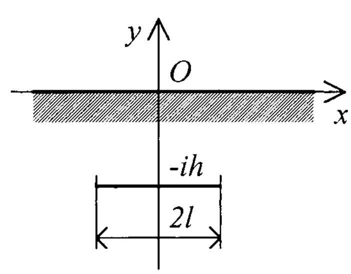

Fig. 4. Crack situated far from the boundary.

The next step consists of reflecting this field from the free boundary. This means that the crack-induced stresses, $\Delta\sigma_y$ and $\Delta\tau_{xy}$, on the line of the future semi-plane boundary ($y = 0$) have to be applied with inverse sign to the boundary of a half-plane without the crack. The potentials for a half-plane, the boundary of which is loaded by normal σ_N and shear σ_T tractions, have the form (e.g., Muskhelishvili, 1963)

$$\Phi_{\mathrm{r}}(z) = -\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\sigma_N - \mathrm{i}\sigma_T}{t - z}\,\mathrm{d}t$$

$$\Psi_{\mathrm{r}}(z) = -\frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\sigma_N + \mathrm{i}\sigma_T}{t - z}\,\mathrm{d}t + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{\infty} \frac{\sigma_N - \mathrm{i}\sigma_T}{(t - z)^2}t\,\mathrm{d}t. \tag{35}$$

By assuming $\sigma_N = -\Delta\sigma_y|_{y=0}$ and $\sigma_T = -\Delta\tau_{xy}|_{y=0}$ and substituting this into Eq. (35), the potentials can be obtained for the reflected stress at the crack line

$$\Phi_{\mathrm{r}} = \frac{\bar{D}_1 - \bar{D}_2}{(z - \mathrm{i}h)^2} + \frac{4\mathrm{i}h\bar{D}_1}{(z - \mathrm{i}h)^3} + O\left(\frac{l^3}{h^3}\right)$$

$$\Psi_{\mathrm{r}} = \frac{\bar{D}_2}{(z - \mathrm{i}h)^2} - \frac{\mathrm{i}h(2\bar{D}_2 - 10\bar{D}_1)}{(z - \mathrm{i}h)^3} - \frac{4h^2\bar{D}_1}{(z - \mathrm{i}h)^4} + O\left(\frac{l^3}{h^3}\right). \tag{36}$$

Variations of these potentials over the crack length are of the order of $O(l/h)^3$. Therefore, with the adopted accuracy they can be neglected. This means that the main asymptotic term of the effect of the crack–boundary interaction can be obtained by assuming the crack to be additionally subjected to uniform stresses (effective stresses, e.g., Chudnovsky and Kachanov, 1983) which are equal to

$$\sigma_{\mathrm{r}} = \frac{3}{h^2}\,\mathrm{Re}\,D_1 + O\left(\frac{l^3}{h^3}\right)$$

$$\tau_{\mathrm{r}} = -\frac{1}{h^2}\,\mathrm{Im}\,D_2 + O\left(\frac{l^3}{h^3}\right). \tag{37}$$

As a result, the SIFs are obtained by adding these tractions to $\sigma_0(t) + \mathrm{i}\tau_0(t)$ and considering the crack as being located in the infinite plane:

$$K_{\mathrm{I}} + \mathrm{i}K_{\mathrm{II}} = -\frac{1}{\sqrt{\pi L}}\left[\int_{-l}^{l}(\sigma_0(x) + \sigma_{\mathrm{r}} + \mathrm{i}\tau_0(x) + \mathrm{i}\tau_{\mathrm{r}})\sqrt{\frac{L + x}{L - x}}\,\mathrm{d}x + O\left(\frac{l^3}{h^3}\right)\right]. \tag{38}$$

The area of the crack opening can be calculated with the same accuracy by considering the crack under the field $\sigma(x) = \sigma_0(x) + \sigma_{\mathrm{r}}$ (e.g., Dyskin and Salganik, 1987); for the plane stress approximation, it has the form

$$S = \frac{4}{E}\int_{-l}^{l}\sqrt{l^2 - x^2}\,\sigma(x)\mathrm{d}x + O\left(\frac{l^3}{h^3}\right). \tag{39}$$

Then

$$S = S_0 + \frac{8}{E}\left[\frac{3\pi}{4}\frac{l^2}{h^2}\,\mathrm{Re}\,D_1 + O\left(\frac{l^3}{h^3}\right)\right], \tag{40}$$

where S_0 is the opening area without the influence of the boundary.

If the initial load is given by a pair of concentrated forces, P [Fig. 3(a)], then

$$\sigma_\mathrm{r} = \frac{3P}{2\pi}\frac{l}{h^2},$$

$$D_1 = \frac{Pl}{2\pi} \tag{41}$$

and, according to Eqs. (38) and (40), the expressions for the SIFs and area of the crack opening have the form

$$K_\mathrm{I} = \frac{P}{\sqrt{\pi l}}\left[1 + \frac{3}{2}\left(\frac{l}{h}\right)^2 + O\left(\frac{l}{h}\right)^3\right],$$

$$K_\mathrm{II} = 0 + O\left(\frac{l}{h}\right)^3$$

$$S = Pl\frac{4}{E}\left[1 + \frac{3}{4}\frac{l^2}{h^2} + O\left(\frac{l^3}{h^3}\right)\right]. \tag{42}$$

If the initial stress is a uniform normal pressure, p [Fig. 3(b)], applied to the crack surfaces, then

$$\sigma_\mathrm{r} = \frac{3p}{4}\frac{l^2}{h^2}$$

$$D_1 = \frac{pl^2}{4}. \tag{43}$$

Accordingly, the expressions for the SIFs and area of the crack opening have the form

$$K_\mathrm{I} = P\sqrt{\pi l}\left[1 + \frac{3}{4}\left(\frac{l}{h}\right)^2 + O\left(\frac{l}{h}\right)^3\right],$$

$$K_\mathrm{II} = 0 + O\left(\frac{l}{h}\right)^3$$

$$S = Pl^2\frac{2\pi}{E}\left[1 + \frac{3}{4}\frac{l^2}{h^2} + O\left(\frac{l^3}{h^3}\right)\right]. \tag{44}$$

Chudnovsky and Kachanov (1983) followed by Kachanov (1987) considered the interaction of cracks taking into account the non-uniformity of the supplementary stresses over the crack line. This allows calculating the next asymptotic term which is beyond the accuracy adopted in this work.

4. Interpolation between two asymptotics. Comparison with numerical solutions

The considered asymptotic solutions are valid for the cracks situated very close to and very far from the boundary. It is natural to obtain general expressions for the stress intensity factors by interpolation between these two extreme cases. The interpolating formula can be derived using the following idea. The result from dipole asymptotics is multiplied by the function $1/(1+al^n)$ (it is assumed that the lengths are normalised by h, i.e. that $h = 1$) that approaches unity as $l \to 0$ and approaches zero as $l \to \infty$. The result from beam asymptotics is multiplied by the function $al^n/(1+al^n)$ that approaches zero as $l \to 0$ and approaches unity as $l \to \infty$. Therefore, both asymptotics contribute exclusively within the domains of their validity. This gives the following interpolating formula:

$$K_{\mathrm{i}} = K_{\mathrm{i}0} + \frac{\Delta K_{\mathrm{i}}^{\mathrm{dip}} + a\Delta K_{\mathrm{i}}^{\mathrm{beam}} l^n}{1 + al^n}, \tag{45}$$

where $K_{\mathrm{i}0}$ is the SIF for the particular loaded crack in an infinite body, $\Delta K_{\mathrm{i}}^{\mathrm{dip}} + K_{\mathrm{i}0}$ and $\Delta K_{\mathrm{i}}^{\mathrm{beam}}$ are the SIFs for the dipole and the first term of the beam asymptotics, respectively, a and n are parameters. For $l \gg 1$ and $l \ll 1$, Eq. (45) is asymptotically equal to the beam and dipole asymptotics, respectively. Here, only supplemental parts of the SIFs are interpolated, reflecting the fact that the initial stresses do not change due to interaction with the surface and contribute everywhere.

Parameter a in Eq. (45) characterises the location of the domain where one asymptote is transferred into the other; parameter n characterises the width of the domains. These parameters will be determined by fitting Eq. (45) to the results of numerical calculations found in the literature. The obtained values of these parameters will be shown to be consistent for all considered cases, indicating that the width and the location of the domains remain the same for all considered examples.

For the case of crack loaded by a pair of concentrated forces, the interpolation formulae for both modes of SIF have the form

$$\frac{K_{\mathrm{I}}}{P} = \frac{1}{\sqrt{\pi l}} + \frac{1}{\sqrt{l}}\frac{Fl^2 + Gal^3}{1 + al^{1.5}}$$

$$\frac{K_{\mathrm{II}}}{P} = \frac{Hal^{2.5}}{1 + al^{1.5}}, \tag{46}$$

where constants $F = 0.5/\sqrt{\pi}$, $G = 0.453$, $H = 0.377$ are set according to Eqs. (27) and (42) applied to the case $h = 1$. The values of parameters, $a = 1.78$ (for the first mode), 1.81 (for the second mode) and $n = 1.5$, were determined by the least square method matching of Eq. (46) to the numerical data of Germanovich and Grekov (1998). The comparison is shown in Fig. 5 (note, the dipole asymptotics contribute nothing into K_{II}). Relative errors are less than 1.5% for the first mode of the SIF and 2.2% for the second mode.

For a uniformly loaded crack, the interpolation formulae assume the form

$$\frac{K_{\mathrm{I}}}{P} = \sqrt{\pi l} + \sqrt{l}\frac{Fl^2 + Gal^3}{1 + al^{1.5}}$$

$$\frac{K_{\mathrm{II}}}{P} = \frac{Hal^{3.5}}{1 + al^{1.5}}, \tag{47}$$

where $F = 3\sqrt{\pi}/4$, $G = 0.483$; $H = 0.377$. Fig. 6 shows comparison of Eq. (47) with the numerical

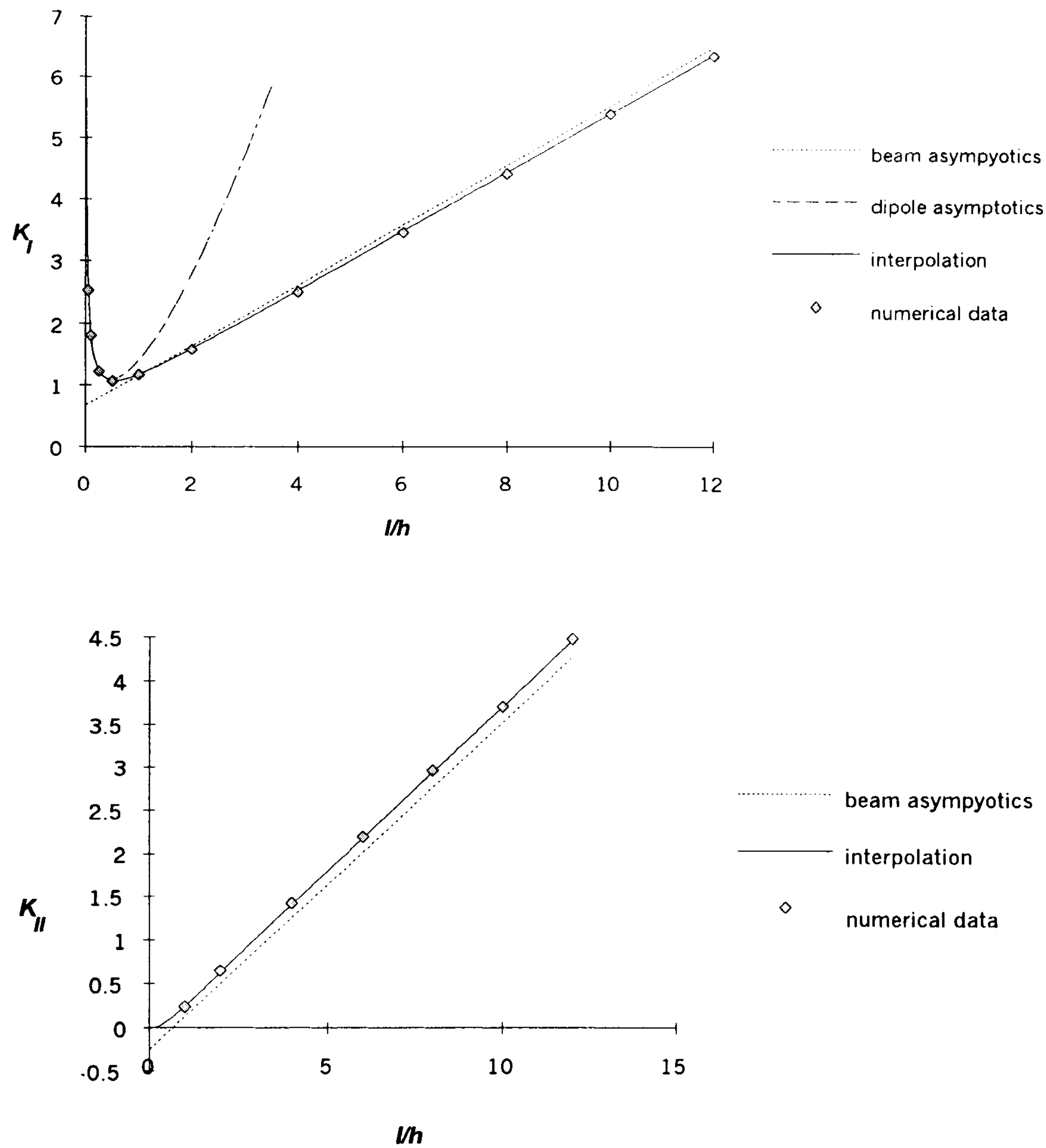

Fig. 5. Comparison with numerical data for a crack loaded by a pair of concentrated forces.

results by Erdogan et al. (1973) and Higashida and Kamada (1982); see also Murakami (1987). One can see that the numerical results from the different sources do not coincide; neither do different handbook descriptions based on the original publications (compare Tada et al., 1985, who use the results by Erdogan et al., 1973, with Savruk, 1981, who utilizes the results by Ashbaugh, 1975). The comparison with the result given by the beam asymptotic approximation allows us to make the choice; indeed, only Higashida and Kamada's solution is in good agreement with the asymptotics (Higashida and Kamada,

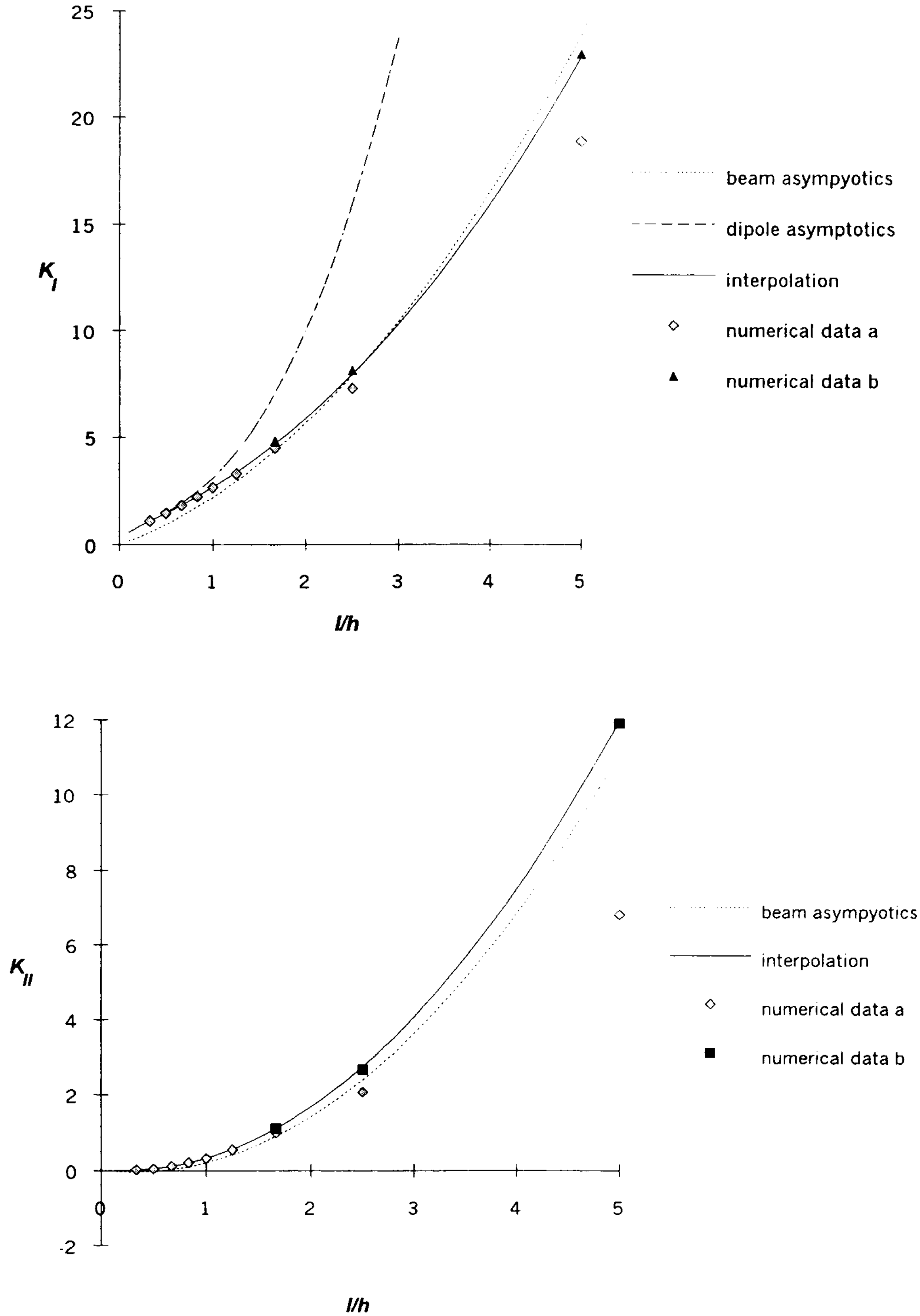

Fig. 6. Comparison with numerical data for a uniformly loaded crack.

1982). The constants determined from matching Eq. (47) to this solution are $a = 1.8$ and $n = 1.5$, which are close to the values obtained for the case of concentrated forces.

5. Strip with a central crack

5.1. Beam asymptotics

The analogue of Eq. (2) for the semi-infinite crack in the strip under normal loading can be obtained from the results of Entov and Salganik (1965) and Foote and Buchwald (1985).

$$K_{\mathrm{I}} = 2\sqrt{3}h^{-3/2}M + 2 \cdot 0.6731\sqrt{3}h^{-1/2}N + O(\mathrm{e}^{-l_{\min}/h}), \tag{48}$$

where M and N are the bending moment and total force applied at the upper layer (above the crack); $l_{\min}$ is the minimum distance from the point of loading.

For the general distribution of normal loading, the direct application of the results from Appendix A concerning the layer between the crack and the boundary shows that the moment, M, and total force, N, fully determine the first two asymptotic terms for the SIF (due to symmetry, in the absence of shear loading, only the mode I SIF is not zero).

For the shear loading, the relation between the main asymptotic term of the second mode of the SIF and the shear force T can be obtained as follows. Consider an infinite beam with a central semi-infinite crack loaded by forces applied as shown in Fig. 7(a). Since in the beam theory, the forces are assumed to be applied to the neutral axis of the beam, as shown in Fig. 7(b), the bending moment $Th/2$ should be additionally applied at the crack tip to compensate the shift of the loads. The potential energy of the whole strip, which is double the energy of the beam, includes the energy of longitudinal deformation and the energy of bending:

$$U = \frac{T^2 l}{Eh} + \frac{3T^2 l}{Eh} = \frac{4T^2 l}{Eh}. \tag{49}$$

The energy release rate is

$$\frac{\mathrm{d}U}{\mathrm{d}l} = \frac{4T^2}{Eh}. \tag{50}$$

Owing to symmetry, only the second mode of the SIF is non-zero in this case, therefore

$$\frac{\mathrm{d}U}{\mathrm{d}l} = \frac{K_{\mathrm{II}}^2}{E}. \tag{51}$$

The comparison of Eqs. (50) and (51) gives

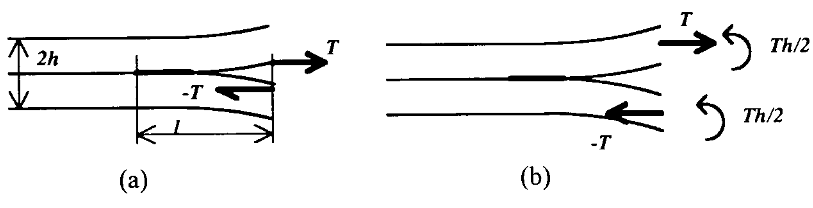

Fig. 7. Crack in a strip under pure shear.

$$K_{\rm II} = 2Th^{-1/2}. \tag{52}$$

Due to linearity, an arbitrary stress distribution can be considered as a superposition of the concentrated forces, thus Eq. (52) remains applicable for any shear loading with the total T; it follows from Appendix A that the details of the distribution affect only terms of higher order than in Eq. (52).

After collecting the results together, both modes of the SIF and the energy release rate can be expressed in the following form

$$\begin{pmatrix} K_{\rm I} \\ K_{\rm II} \end{pmatrix} = \begin{pmatrix} 2\sqrt{3} \\ 0 \end{pmatrix} h^{-3/2} M + \begin{pmatrix} 2.332 \\ 0 \end{pmatrix} h^{-1/2} N + \begin{pmatrix} 0 \\ 2 \end{pmatrix} h^{-1/2} T$$

$$\frac{{\rm d}U}{{\rm d}l} = \frac{12(M + 0.6731Nh)^2}{Eh^3} + \frac{4T^2}{Eh}. \tag{53}$$

By repeating the procedure of matching the asymptotics, described in Section 2, it can be shown that the total force, N, and moment, M, can be found by considering the corresponding clamped beam with the coefficient of elastic clamping

$$\frac{k'}{Eh^2} = \frac{h\delta'}{EI}, \; \delta' = 0.673. \tag{54}$$

It has been taken into account that, in the case of the strip, both faces of the crack correspond to beams and contribute equally to the main term.

For pure tensile cracks, the final expression has the form

$$K_{\rm I} = 2\sqrt{3}M_0\left(1 - \delta'\frac{h}{l}\right)h^{-3/2} + 2\sqrt{3}\delta Nh^{-1/2}, \tag{55}$$

where M_0 is the bending moment calculated for the purely clamped beam.

5.2. Examples of symmetrically loaded cracks

Consider a strip of thickness $2h$ with a symmetrically located (central) parallel crack of length $2l$. Three types of loading will be considered: loading with the pair of normal concentrated forces P applied at the crack centre [Fig. 8(a)], uniform loading with the normal stress p [Fig. 8(b)], uniform loading with a shear stress τ [Fig. 8(c)]. Due to symmetry, only the first modes of SIF are present in the first two cases and only the second mode is present in the third case.

By substituting Eq. (26) or Eq. (29) into Eq. (55) (similar to Section 2.5), the SIF for loading by a pair of concentrated forces [Fig. 8(a)] and for uniform loading [Fig. 8(b)] can be found in the following form:

$$K_{\rm I} = P\frac{\sqrt{3}}{2}\frac{l}{h^{3/2}}\left(1 + \delta'\frac{h}{l} + O\left(\frac{h^2}{l^2}\right)\right) \text{ for Fig. 8(a)}, \tag{56}$$

$$K_{\rm I} = p\frac{2\sqrt{3}}{3}\frac{l^2}{h^{3/2}}\left(1 + 2\delta'\frac{h}{l} + O\left(\frac{h^2}{l^2}\right)\right) \text{ for Fig. 8(b)}. \tag{57}$$

The main asymptotic terms in Eqs. (56) and (57) may also be obtained by considering the energy balance of the clamped beams (e.g. Rice, 1968a, 1968b; Slepian, 1981).

For the case of uniform shear loading, τ [Fig. 8(c)], the SIF is

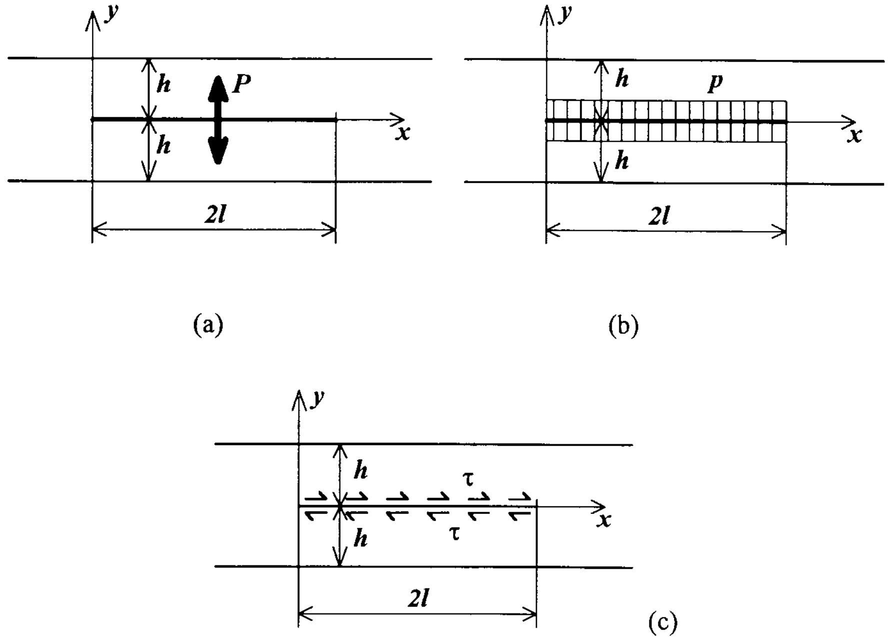

Fig. 8. Central crack in a strip.

$$K_{\mathrm{II}} = 2\tau \frac{l}{h^{1/2}}\left(1 + O\left(\frac{h}{l}\right)\right). \tag{58}$$

The comparison of these asymptotic solutions with the numerical results of Cinar and Erdogan (1983) are presented in Figs. 9–11.

The area of the crack opening can be calculated the same way as for the cracks in half-plane. However, here the displacements of both upper and lower faces correspond to deflections of the beams. Therefore, the area of the crack opening for these cases should be calculated by doubling the corresponding results for the half-plane problems. For the third case, the area of the crack opening (the first mode opening, to be precise) is obviously equal to zero.

5.3. Dipole asymptotics

When the crack length is much less than the strip width, the dipole asymptotic solution (e.g. Savruk, 1981) gives

$$K_{\mathrm{I}}^{\mathrm{dip}} = \frac{P}{\sqrt{\pi l}}\left[1 + 2.2836\frac{l^2}{h^2} + O\left(\frac{l^4}{h^4}\right)\right] \text{ for Fig. 8(a),} \tag{59}$$

$$K_{\mathrm{I}}^{\mathrm{dip}} = p\sqrt{\pi l}\left[1 + 1.1417\frac{l^2}{h^2} + O\left(\frac{l^4}{h^4}\right)\right] \text{ for Fig. 8(b)} \tag{60}$$

$$K_{\mathrm{II}}^{\mathrm{dip}} = \tau\sqrt{\pi l}\left[1 + 0.6668\frac{l^2}{h^2} + O\left(\frac{l^4}{h^4}\right)\right] \text{ for Fig. 8(c).} \tag{61}$$

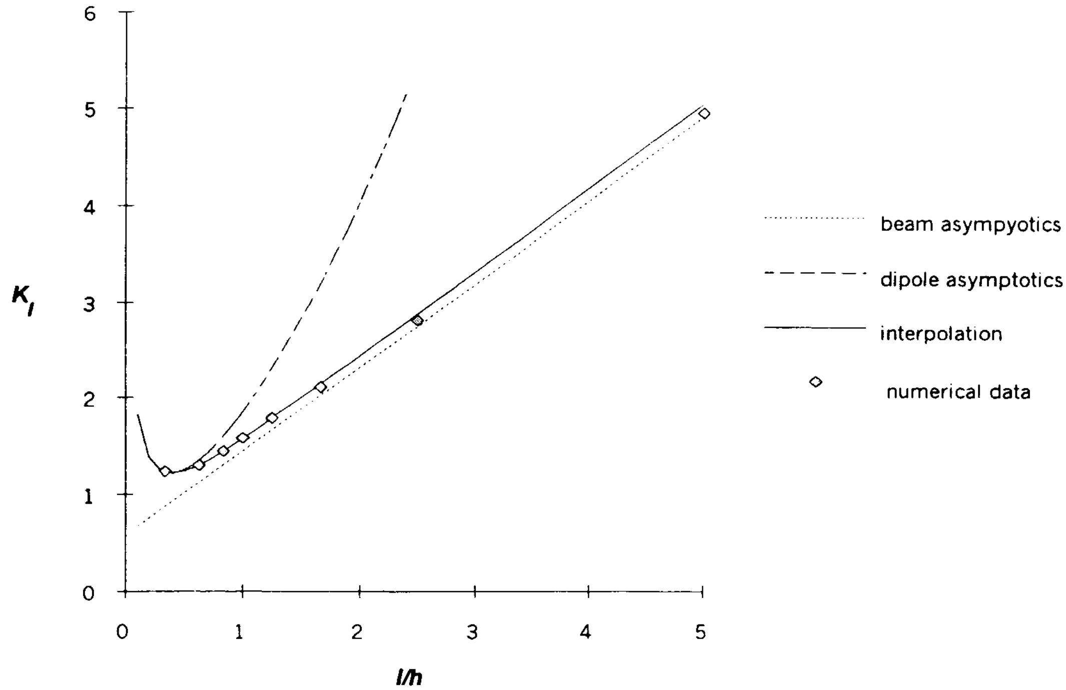

Fig. 9. Comparison with numerical data for a crack loaded by a pair of concentrated forces.

The areas of the crack openings can be calculated similarly to the case of the half-plane. The results are:

$$S^{\text{dip}} = P\frac{8}{E}l\left[0.5 + 0.570\frac{l^2}{h^2} + O\left(\frac{l^4}{h^4}\right)\right] \text{ for Fig. 8(a)} \tag{62}$$

$$S^{\text{dip}} = p\frac{8}{E}l^2\left[\frac{\pi}{4} + 0.895\frac{l^2}{h^2} + O\left(\frac{l^4}{h^4}\right)\right] \text{ for Fig. 8(b).} \tag{63}$$

For the third case, the area is evidently of the order neglected in this approximation.

5.4. Interpolation

The interpolation in the form of Eq. (45) may be suggested to extend the obtained asymptotic solutions for arbitrary values of l/h. For the problem from Fig. 8(a), the interpolating formula is the same as the one for the first mode of the SIF from Eq. (46), the constants being $F = 2.2836/\sqrt{\pi}$ and $G=\sqrt{3}/2$. The parameters $a = 1.84$ and $n = 1.5$ were determined by the least square method so that the formulae fit the numerical data of Cinar and Erdogan (1983).

For the problem from Fig. 8(b), the interpolating formula for K_{I} has the form of Eq. (47), the constants being $F = 0.147\sqrt{\pi}$, $G = 2\sqrt{3}/3$, $a = 1.808$ and $n = 1.5$. The comparison with the numerical results of Cinar and Erdogan (1983) is presented in Figs. 9–11. The interpolation for the case of shear [Fig. 8(c)] were not conducted since the beam asymptotics coincided relatively well with the available numerical data for all values of l/h (see Fig. 11).

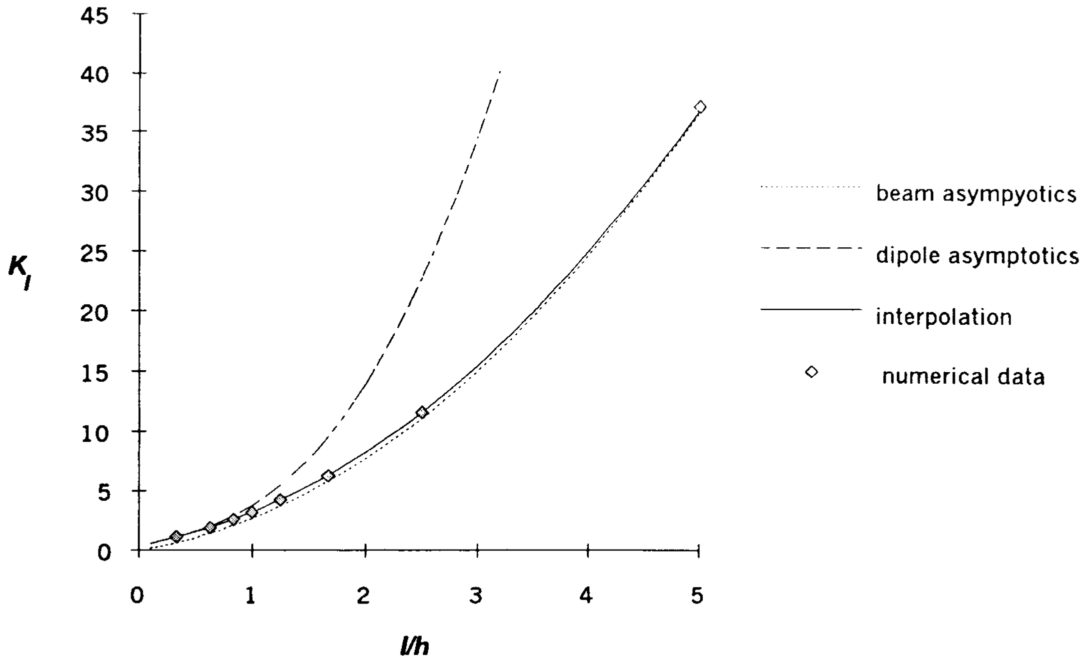

Fig. 10. Comparison with numerical data for a uniformly loaded crack in a strip.

6. Conclusion

The obtained asymptotic solutions provide a powerful tool for analysing the interaction between a crack and one or two parallel free boundaries. The beam approximation accounts for small distances between the crack and the boundary, while the dipole asymptotics accounts for large distances.

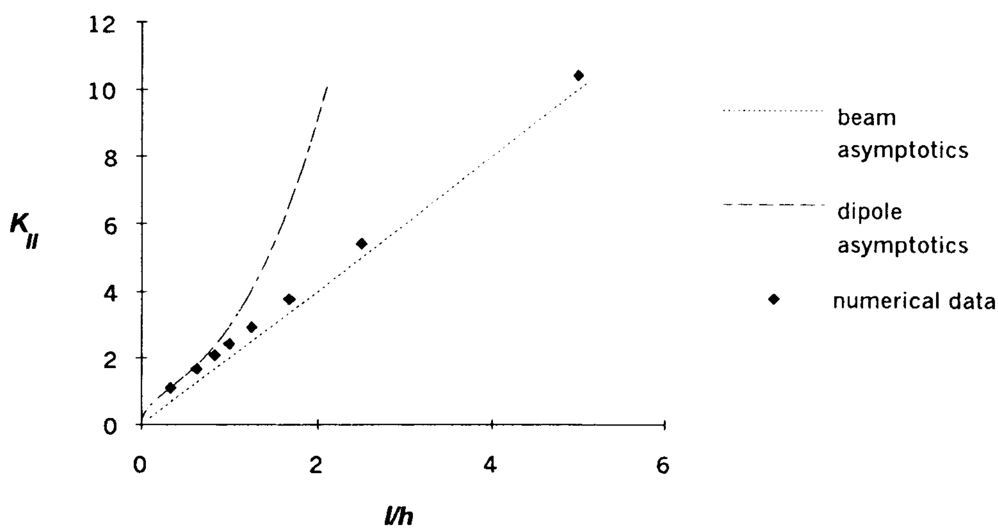

Fig. 11. Comparison with numerical data for the crack in a strip under shear.

Table 1
The obtained asymptotic and interpolating formulae for SIFs. Whenever the values of K_{II} are relevant, they are presented in brackets

Configuration	K_1/P, or K_1/p		
	Dipole	Beam	Interpolation
Strip, pair of concentrated forces	$\frac{1}{\sqrt{\pi l}}+1.289\, l^{1.5}$	$\frac{\sqrt{3}}{2}l$	$\frac{1}{\sqrt{\pi l}}+\frac{1}{\sqrt{l}}\frac{Al^2+Bal^3}{1+al^{1.5}}$
			$A=1.289;\ B=\frac{\sqrt{3}}{2};\ a=1.84$
Strip, uniform normal loading	$\sqrt{\pi l}+2.023l^{2.5}$	$\frac{2\sqrt{3}}{3}l^2$	$\sqrt{\pi l}+\sqrt{l}\frac{Al^2+Bal^3}{1+al^{1.5}}$
			$A=2.032;\ B=\frac{2\sqrt{3}}{3};\ a=1.808$
Half-plane, pair of concentrated forces	$\frac{1}{\sqrt{\pi l}}+\frac{3}{2\sqrt{\pi}}l^{1.5}$	$0.483l$ $K_{\text{II}}=0.337l$	$\frac{1}{\sqrt{\pi l}}+\frac{1}{\sqrt{l}}\frac{Al^2+Bal^3}{1+al^{1.5}}$
			$A=\frac{3}{2\sqrt{\pi}};\ B=0.483;\ C=0.377;\ a=1.78$
			$\left(K_{\text{II}}=\frac{Cal^{2.5}}{1+al^{1.5}},\ a=1.81\right)$
Half-plane, uniform normal loading	$\sqrt{\pi l}+\frac{3\sqrt{\pi}}{4}l^{2.5}$	$0.644\, l^2$ $(K_{\text{II}}=0.502\, l^2)$	$\sqrt{\pi l}+\sqrt{l}\frac{Al^2+Bal^3}{1+al^{1.5}}$
			$A=\frac{3\sqrt{\pi}}{4};\ B=0.644;\ C=0.502;\ a=1.8$
			$\left(K_{\text{II}}=\frac{Cal^{3.5}}{1+al^{1.5}},\ a=1.8\right)$

Nevertheless, the comparison with various numerical results suggests that the ranges of applicability of these two asymptotics are rather wide. For intermediate distances a simple two-parametric interpolating formula allows covering the whole range of the distances and gives results which correspond well to the numerical data. It is interesting that the values obtained for the parameters of the interpolating formulae by fitting to the numerical data are consistent for different geometries and loads.

All the results including the interpolating[4] formulae for the above value of parameter n are collected in Tables 1 and 2. For all the examples, it is supposed that $h = 1$.

[4] The interpolation was conducted only for the cases when the corresponding numerical results were available. That is why interpolating formulae for the area of the crack opening are not provided.

Table 2
The obtained asymptotic formulae for the area of crack opening

Configuration	Area $\frac{S}{P}\frac{\kappa+1}{\mu}$	
	Dipole	Beam
Strip, pair of concentrated forces	$0.5\,l + 0.570\,l^3$	$1/8\,l^4$
Strip, uniform loading	$\pi/4\,l^2 + 0.895\,l^4$	$2/15\,l^5$
Half-plane, pair of concentrated forces	$1/2\,l + 0.375\,l^3$	$1/16\,l^4$
Half-plane, uniform loading	$\pi/4\,l^2 + 3\pi/16\,l^4$	$1/15\,l^5$

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Appendix A. Contribution of load with zero total and moment into the stress intensity factors

Let the semi-infinite crack occupy the part $x > 0$ of the x-axis [Fig. 1(c)]. According to the solution by Zlatin and Khrapkov (1986), the stress intensity factors, $K_{\rm I}$ and $K_{\rm II}$, and the energy release rate due to crack propagation are given by Eqs. (2) and (3). In this solution, the crack faces are supposed to be free of load; hence, in this sense, it is a solution of the homogeneous problem chosen in such a way that the given total force and moment are produced by the stress distribution on the crack continuation line, $x > 0$, $y = 0$ [Fig. 1(c)]. In order to be able to use Eqs. (2) and (3) for a non-homogeneous problem, i.e., for the semi-infinite crack loaded within the interval $(0, l)$, it is necessary to show that Eqs. (2) and (3) correctly give the first two asymptotic terms with respect to $l/h \to \infty$. In other words, varying load distributions while keeping the same total force and moment should affect only terms of the order of $Nh^{-1/2}o(1)$.

We will begin the proof with a particular case of a concentrated force (N, T) applied at a distance $l \gg h$ from the crack tip (Fig. A1). It will be shown that

$$\begin{pmatrix} K_{\rm I} \\ K_{\rm II} \end{pmatrix} h^{1/2} = \left[K_M \frac{l}{h} + K_N + o(1)\right] N + [K_T + o(1)]T \text{ as } \frac{l}{h} \to \infty. \tag{A1}$$

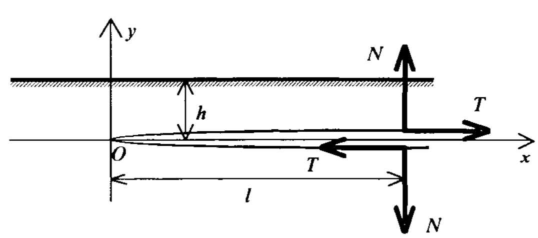

Fig. A1. Crack loaded by a pair of concentrated forces.

Indeed, since the problem is linear, for the normal force, N, the SIFs can be written as follows

$$\begin{pmatrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{pmatrix} h^{1/2} = \left[K_M \frac{l}{h} + K_N + f_N\left(\frac{l}{h}\right) \right] N + \left[K_T + f_T\left(\frac{l}{h}\right) \right] T, \tag{A2}$$

where $f_N(l/h)$ and $f_T(l/h)$ are some continuous functions yet to be estimated. However, according to Eq. (2),

$$\lim_{l \to \infty} \begin{pmatrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{pmatrix} h^{1/2} = \left[K_M \frac{l}{h} + K_N \right] N + K_T T. \tag{A3}$$

Hence,

$$f_N\left(\frac{l}{h}\right) \to 0,$$

$$f_T\left(\frac{l}{h}\right) \to 0 \text{ as } \frac{l}{h} \to \infty \tag{A4}$$

which proves Eq. (A1).

Note that expression (A1) was written for $l \to \infty$, h = const., but can also be looked upon as being valid for $h \to 0$, l = const. Therefore, the above result is also valid for any number of concentrated forces, as long as there exists a minimum distance, l_{min}, so that a force cannot be applied closer to the crack tip. Then, the error of the asymptotics, Eqs. (2) and (3), vanishes as $h/l_{\mathrm{min}} \to 0$.

Consider now the case of a continuous load distribution, $\sigma(x)$, $\tau(x)$, $0 \le x \le l$, where l is fixed and $\sigma(x)$ and $\tau(x)$ are independent of h. Then, the case of a distributed load can be analyzed using Eq. (A2) as a Green function:

$$\begin{pmatrix} K_{\mathrm{I}} \\ K_{\mathrm{II}} \end{pmatrix} h^{1/2} = K_M \frac{M}{h} + K_N N + K_T T + \int_0^l f_N\left(\frac{x}{h}\right) \sigma(x) \mathrm{d}x + \int_0^l f_T\left(\frac{x}{h}\right) \tau(x) \mathrm{d}x, \tag{A5}$$

where

$$M = \int_0^l x \sigma(x) \mathrm{d}x,$$

$$(N, T) = \int_0^l (\sigma(x), \tau(x)) \mathrm{d}x. \tag{A6}$$

By introducing a $\xi \in (0, l/h)$, the first integral in Eq. (A5) can be written in the following form:

$$I_N(h, l) = \int_0^l f_N\left(\frac{x}{h}\right) \sigma(x) \mathrm{d}x = \int_0^{\xi h} f_N\left(\frac{x}{h}\right) \sigma(x) \mathrm{d}x + \int_{\xi h}^l f_N\left(\frac{x}{h}\right) \sigma(x) \mathrm{d}x. \tag{A7}$$

The last integral in Eq. (A7) can be evaluated for $h \to 0$ as follows:

$$\left| \int_{\xi h}^l f_N\left(\frac{x}{h}\right) \sigma(x) \mathrm{d}x \right| \le \int_{\xi h}^l \left| f_N\left(\frac{x}{h}\right) \right| |\sigma(x)| \mathrm{d}x \le \max_{[\xi h, l]} \left| f_N\left(\frac{x}{h}\right) \right| \left| \int_{\xi h}^l |\sigma(x)| \mathrm{d}x \le \max_{[\xi h, l]} \left| f_N\left(\frac{x}{h}\right) \right| \right| \int_0^l |\sigma(x)| \mathrm{d}x. \tag{A8}$$

Suppose that the absolute value of the load $\sigma(x)$ is integrable. Then for any given $\varepsilon > 0$, the integral can be made less than $\varepsilon/2$ by choosing sufficiently large ξ.

Now consider the first integral in Eq. (A7) for the chosen value of ξ. Suppose $\sigma(x) \sim \sigma_0$ as $x \to 0$. Then

$$\int_0^{\xi h} f_N\left(\frac{x}{h}\right)\sigma(x)\mathrm{d}x = h\int_0^{\xi} f_N(t)\sigma(th)\mathrm{d}t \sim h\sigma_0\int_0^{\xi} f_N(t)\mathrm{d}t \text{ for } h\to 0, \tag{A9}$$

provided that the integral

$$I_0 = \sigma_0\int_0^{\xi} f_N(t)\mathrm{d}t$$

exist. According to Eq. (A5), this integral is the difference between the exact and the asymptotic solution Eq. (2) for SIFs of a semi-infinite crack situated at a unit distance from the free boundary and subjected to uniform load σ_0 over the fixed length ξ. The exact solution for a *semi-infinite* crack is, in fact, an asymptotic one for the solution for a long *finite* crack of length $l_f \gg \xi$, uniformly loaded at length ξ. The latter does exists (e.g. Savruk, 1981) and therefore, I_0 exist as well. Hence, $I_0 < \infty$ and Eq. (A9) can be made less than $\varepsilon/2$ by choosing a sufficiently small h. Thus,

$$I_N \to 0 \text{ as } \frac{l}{h} \to \infty. \tag{A10}$$

Similarly,

$$I_T = \int_0^{l} f_T\left(\frac{x}{h}\right)\tau(x)\mathrm{d}s \text{ as } \frac{l}{h} \to \infty. \tag{A11}$$

Therefore, Eqs. (2) and (3) also correctly give the two successive main asymptotic terms for the distributed load $\sigma(x)$, $\tau(x)$, provided that

$$\int_0^{l} |\sigma(x)|\mathrm{d}x < \infty,$$

$$\int_0^{l} |\tau(x)|\mathrm{d}x < \infty,$$

$$\sigma(x) \to \sigma_0 < \infty,$$

$$\tau(x) \to \tau_0 < \infty \text{ as } \frac{l}{h} \to \infty. \tag{A12}$$

It should be noted that if both the total force and moment of the applied load are zero, this asymptotic representation does not make sense, and Eqs. (2) and (3) cannot be used. Otherwise, it provides a justification of the beam approximation (e.g., Rice, 1968a, 1968b; Slepian, 1981; Williams, 1988; Bolotin, 1996), according to which, it is possible to apply the beam theory (i.e., the assumption of linear stress distribution through the beam length) even at the crack-tip region with high stress concentration (see, for example, the work by Williams, 1988).

Appendix B. Matching the energy release rates

It will be shown here that the direct matching of the energy release rates computed from the elementary beam solution and the SIFs (necessary for the determination of the elastic clamping coefficient, k) is possible with the adopted accuracy. In the case of loading by concentrated forces, N (Fig. 2), it is therefore necessary to show that

$$U = 2\frac{N^2}{E}\left(\frac{l}{h}\right)^3\left[1+\frac{k}{4}\frac{h}{l}+\frac{h}{l}\alpha\left(\frac{h}{l}\right)\right],\ \alpha\left(\frac{h}{l}\right)\to 0,\ \left(\frac{h}{l}\to 0\right). \tag{B1}$$

Here the term $(l/h)^2\alpha(h/l)$ includes the work done by the concentrated forces on the displacements of the crack faces, which are not accounted for by the elementary beam solution. These displacements are:

1. the correction to the neutral axis displacement which consists of a term $O(l/h)$ and exponentially vanishing terms (Timoshenko and Goodier, 1970),
2. global displacement of the lower face of the crack which, for the case of concentrated force applied at a distance l, is of the order of $\ln(l)$, and
3. local displacements at the points of application of the load which do not depend on l.

This means that indeed in Eq. (B1), $\alpha(h/l)\to 0$ as $h/l\to 0$.

Accordingly, the energy release rate is

$$J = \frac{dU}{dl} = \frac{N^2l^2}{Eh^3}\left[6+k\frac{h}{l}+\frac{h}{l}\alpha\left(\frac{h}{l}\right)-\left(\frac{h}{l}\right)^2\alpha'\left(\frac{h}{l}\right)\right]. \tag{B2}$$

Comparing Eq. (B2) with the energy release rate computed using SIFs [see Eq. (3)],

$$J = \frac{dU}{dl} = \frac{N^2l^2}{Eh^3}\left[6+12\delta\frac{h}{l}+o\left(\frac{h}{l}\right)\right]\ \text{as}\ \frac{h}{l}\to 0, \tag{B3}$$

we arrive at Eq. (19).

Appendix C. Contribution of the lower face displacement into the area of crack opening

The displacement of the lower crack face can be estimated by considering a half-plane with the step boundary BDOE (Fig. C1), loaded by the applied normal load, $\sigma(x)$, acting over length l of part OE and an additional load, $(\sigma_c(y),\ \tau_c(y))$ which replaces the action of the removed part, EOBC.

The leading asymptotic term for the displacement caused by load $\sigma(x)$ is obviously given by the corresponding displacement of a half-plane with straight boundary half of which is loaded by $\sigma(x)$ and another half is free. This displacement is of the order of $\bar{\sigma}$ as $h/l\to 0$, where $\bar{\sigma}=\langle\sigma(x)\rangle$ is the average stress.

The leading terms of the part of the displacement caused by the additional load, $(\sigma_c(y),\ \tau_c(y))$, should coincide with the displacement of the straight boundary of the half-plane loaded at the origin by a concentrated force, $(N,\ T)$ and a moment, M, representing this load. Due to equilibrium, $N,T\sim\bar{\sigma}l$, $M\sim\bar{\sigma}l^2$. Therefore, near the crack tip (in a region of the length of order h) the maximum displacement will be of the order of $E^{-1}\bar{\sigma}l^2h^{-1}$. Far from the origin, the displacement will increase as $E^{-1}\bar{\sigma}l\ln xh^{-1}$.

Now the part of the area of the crack opening, S_{lower}, associated with the displacement of the lower face of the crack consists of the contribution of the near-tip region, which is of the order of $E^{-1}\bar{\sigma}l^2$, and the contribution of the logarithmically increasing displacement, which is of the order of $E^{-1}\bar{\sigma}l^2\ln lh^{-1}$.

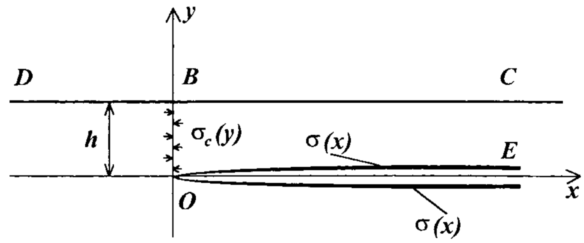

Fig. C1. Semi-infinite crack under load $\sigma(x)$.

As a result,

$$S_{\text{lower}} \sim E^{-1}\bar{\sigma}l^2 \ln \frac{l}{h}. \tag{C1}$$

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